On singular integral operators with semi-almost periodic coefficients on variable Lebesgue spaces

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Abstract

Let a be a semi-almost periodic matrix function with the almost periodic representatives a_l and a_r at $-\infty$ and $+\infty$, respectively. Suppose $p:\mathbb{R}\to(1,\infty)$ is a slowly oscillating exponent such that the Cauchy singular integral operator S is bounded on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R})$. We prove that if the operator $aP+Q$ with $P=(I+S)/2$ and $Q=(I-S)/2$ is Fredholm on the variable Lebesgue space $L_N^{p(\cdot)}(\mathbb{R})$, then the operators a_lP+Q and a_rP+Q are invertible on standard Lebesgue spaces $L_N^{q_l}(\mathbb{R})$ and $L_N^{q_r}(\mathbb{R})$ with some exponents q_l and q_r lying in the segments between the lower and the upper limits of p at $-\infty$ and $+\infty$, respectively.

Keywords: Almost-periodic function, semi-almost periodic function, slowly oscillating function, variable Lebesgue space, singular integral operator, Fredholmness, invertibility

1. Introduction

Given a Banach space X, we denote by X_N the Banach space of all columns of height N with components in X; the norm in X_N is defined by

$$\|(x_1,\dots,x_N)^{\mathrm{T}}\|_{X_N}=\left(\sum_{\alpha=1}^{N}\|x_\alpha\|_X^2\right)^{1/2}.$$

Given a subalgebra B of $L^\infty(\mathbb{R})$, we denote by $B_{N\times N}$ the algebra of all $N\times N$ matrices with entries in B; we equip $B_{N\times N}$ with the norm

$$\|a\|_{B_{N\times N}}=\|(a_{\alpha\beta})_{\alpha,\beta=1}^{N}\|_{B_{N\times N}}=\left(\sum_{\alpha,\beta=1}^{N}\|a_{\alpha\beta}\|_B^2\right)^{1/2}.$$

Let $\mathcal{B}(X)$ denote the Banach algebra of all bounded linear operators on X and let $\mathcal{K}(X)$ denote the ideal of all compact operators on X. As usual, A^* denotes the adjoint operator of $A\in\mathcal{B}(X)$. An operator $A\in\mathcal{B}(X)$ is said to be Fredholm on X if its image $\operatorname{Im}A$ is closed in X and

$$\dim\operatorname{Ker}A<\infty,\quad \dim(X/\overline{\operatorname{Im}A})<\infty.$$

We denote by $C(\overline{\mathbb{R}})$ the set of all complex-valued continuous functions c on $\mathbb{R}$ which have finite limits $c(-\infty)$ and $c(+\infty)$ at $-\infty$ and $+\infty$. Let $C(\dot{\mathbb{R}})$ be the set of all functions $c\in C(\overline{\mathbb{R}})$ such that $c(-\infty)=c(+\infty)$.

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An almost-periodic polynomial is a function of the form

$$a(x) = \sum_{j=1}^{m} a_j e^{i\lambda_j x} \quad (x \in \mathbb{R}) \quad \text{with} \quad a_j \in \mathbb{C}, \quad \lambda_j \in \mathbb{R}.$$

The set of all almost-periodic polynomials will be denoted by AP^0. The algebra AP of the continuous almost-periodic functions is defined as the closure of AP^0 in $L^\infty(\mathbb{R})$; its closure with respect to a stronger *Wiener* norm $\|a\|_W := \sum |a_j|$ is the algebra APW. Note that APW is dense in AP. Finally, the algebra SAP of the semi-almost-periodic functions is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ containing $C(\overline{\mathbb{R}}) \cup AP$. The algebra SAP was introduced by Sarason [37], who also showed that every $a \in SAP_{N\times N}$ can be written in the form

$$a = (1-u)a_l + ua_r + a_0,$$

where $u \in C(\overline{\mathbb{R}})$ is any fixed function such that $0 \le u \le 1$, $u(-\infty) = 0$, $u(+\infty) = 1$, a_l and a_r belong to $AP_{N\times N}$, and a_0 is in $[C_0]_{N\times N}$, the set of all continuous matrix functions vanishing at $-\infty$ and $+\infty$. Moreover, a_l and a_r are uniquely determined by a and the maps $a \mapsto a_l$ and $a \mapsto a_r$ are C^*-algebra homomorphisms of $SAP_{N\times N}$ onto $AP_{N\times N}$. The matrix functions a_l and a_r are referred to as the almost-periodic representatives of a at $-\infty$ and $+\infty$, respectively (for $N = 1$, see [6, Theorem 1.21]; for $N > 1$, the proof is the same).

For a continuous function $f : \mathbb{R} \to \mathbb{C}$ and $J \subset \mathbb{R}$, let

$$\operatorname{osc}(f, J) := \sup_{t,\tau \in J} |f(t) - f(\tau)|.$$

Following [31], we denote by SO the class of slowly oscillating functions given by

$$SO := \left\{ f \in C(\mathbb{R})\colon \lim_{x\to+\infty} \operatorname{osc}(f, [-2x, -x] \cup [x, 2x]) = 0 \right\} \cap L^\infty(\mathbb{R}).$$

Clearly, SO is a unital C^*-subalgebra of $L^\infty(\mathbb{R})$ that contains $C(\dot{\mathbb{R}})$.

Let $p\colon \mathbb{R} \to [1, \infty]$ be a measurable a.e. finite function. By $L^{p(\cdot)}(\mathbb{R})$ we denote the set of all complex-valued functions f on $\mathbb{R}$ such that

$$I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}} |f(x)/\lambda|^{p(x)} dx < \infty$$

for some $\lambda > 0$. This set becomes a Banach space when equipped with the norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : I_{p(\cdot)}(f/\lambda) \le 1 \right\}.$$

It is easy to see that if p is constant, then $L^{p(\cdot)}(\mathbb{R})$ is nothing but the standard Lebesgue space $L^p(\mathbb{R})$. The space $L^{p(\cdot)}(\mathbb{R})$ is referred to as a *variable Lebesgue space*. We will always suppose that

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}} p(x) =: p_+ < \infty. \tag{1.1}$$

Under these conditions, the space $L^{p(\cdot)}(\mathbb{R})$ is separable and reflexive, its dual space is isomorphic to the space $L^{p'(\cdot)}(\mathbb{R})$, where

$$1/p(x) + 1/p'(x) = 1 \quad (x \in \mathbb{R})$$

(see, e.g., [24]).

The Cauchy singular integral operator S is defined for $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ by

$$(Sf)(x) := \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\tau)}{\tau - x} d\tau \quad (x \in \mathbb{R}),$$

where the integral is understood in the principal value sense. Assume that S generates a bounded operator on $L^{p(\cdot)}(\mathbb{R})$ and put

$$P := (I + S)/2, \quad Q := (I - S)/2.$$

The operators S, P, and Q are defined on $L_N^{p(\cdot)}(\mathbb{R})$ elementwise. If $a \in L^\infty_{N\times N}(\mathbb{R})$, then the operator aI of multiplication by a is bounded on $L_N^{p(\cdot)}(\mathbb{R})$. We will say that the operator $aP + Q$ with $a \in L^\infty_{N\times N}(\mathbb{R})$ is a singular integral operator with the coefficient a.

A Fredholm criterion for Banach algebras of singular integral operators with piecewise continuous coefficients on variable Lebesgue spaces $L^{p(\cdot)}(\Gamma, w)$ over Carleson Jordan curves with weights having finite sets of singularities were obtained in [17–19] (see also the references therein). The approach of these works is based on further developments of the methods of the monograph [4] based on localization techniques, Wiener-Hopf factorization and heavy use of results and methods from the theory of submultiplicative functions. An alternative approach to Fredholm theory of singular integral operators with piecewise continuous and slowly oscillating coefficients is based on the method of limit operators and Mellin pseudodifferential operators techniques (we refer to [32], [4, Section 10.6], [5] in the case of standard Lebesgue spaces and to [34, 35] in the case of weighted variable Lebesgue spaces). The second approach allows one to treat the case of composed curves, but still not arbitrary composed Carleson curves.

Notice that in all mentioned works coefficients are piecewise continuous or slowly oscillating; and the variable exponent p is continuous and has a finite limit at infinity in the case of unbounded curves. The aim of the present paper is to make the first step beyond these hypotheses: we are going to study singular integral operators $aP + Q$ with $a \in SAP_{N\times N}$ on variable exponent spaces with the exponent p which may not have a limit at infinity.

Let $\mathcal{E}$ denote the class of exponents $p : \mathbb{R} \to [1, \infty]$ satisfying (1.1), continuous on $\mathbb{R}$, and such that the Cauchy singular integral operator is bounded on $L^{p(\cdot)}(\mathbb{R})$. First, we observe that this class contains interesting exponents.

Lemma 1.1. *There exists an exponent $p \in \mathcal{E}$ such that $p \in SO \setminus C(\dot{\mathbb{R}})$.*

Lerner [27] constructed an example of a variable exponent $p_L \notin C(\dot{\mathbb{R}})$ such that the Hardy-Littlewood maximal operator M is bounded on $L^{p_L(\cdot)}(\mathbb{R})$. It is known that the boundedness of the Hardy-Littlewood maximal operator implies the boundedness of the Cauchy singular integral operator [10, 11, 20]. Thus $p_L \in \mathcal{E}$. It turns out that $p_L \in SO$, which gives the proof of Lemma 1.1. All details of the proof of Lemma 1.1 are contained in Section 2.

Our main result is the following.

Theorem 1.2 (Main result). *Let $a \in SAP_{N\times N}$ and $p \in \mathcal{E} \cap SO$. If the operator $aP + Q$ is Fredholm on the variable Lebesgue space $L_N^{p(\cdot)}(\mathbb{R})$, then*

(a) *there is an exponent q_r lying in the segment*

$$\left[\liminf_{x\to+\infty} p(x), \limsup_{x\to+\infty} p(x)\right]$$

such that $a_r P + Q$ is invertible on the standard Lebesgue space $L_N^{q_r}(\mathbb{R})$.

(b) *there is an exponent q_l lying in the segment*

$$\left[\liminf_{x\to-\infty} p(x), \limsup_{x\to-\infty} p(x)\right]$$

such that $a_l P + Q$ is invertible on the standard Lebesgue space $L_N^{q_l}(\mathbb{R})$.

For standard Lebesgue spaces this result boils down to the statement that Fredholmness of $aP + Q$ with $a \in SAP_{N\times N}$ on $L_N^p(\mathbb{R})$ implies the invertibility of $a_r P + Q$, $a_l P + Q$ on the same space $L_N^p(\mathbb{R})$, and in this form was established in [21] (see also its proof in [6, Chap. 18]).

Note also that if $b \in APW_{N\times N}$, then the operator $bP + Q$ is invertible on all standard Lebesgue spaces $L^p_N(\mathbb{R})$, $1 < p < \infty$, as soon as it is invertible on at least one of them (see [6, Section 18.1]). It is not known at the moment whether this property persists for all $b \in AP_{N\times N}$. In particular, we do not know whether in the setting of Theorem 1.2 the operators $a_l P + Q$ and $a_r P + Q$ are invertible on $L^{q_l}(\mathbb{R})$ and $L^{q_r}(\mathbb{R})$ for all q_l and q_r in the segments between the lower and the upper limits of p at $-\infty$ and $+\infty$, respectively.

The proofs in [6, 21] are based on the method of limit operators. The outline of this method is as follows. Let $h \in \mathbb{R}$ and V_h be the translation operator given by

$$(V_h f)(x) := f(x - h) \quad (x \in \mathbb{R}).$$

It is well known that this operator is an isometry on every standard Lebesgue space. Moreover, it commutes with the Cauchy singular integral operator S. The method of limit operators consists in the study of the strong limits of $V_{-h_k} A V_{h_k}$ as $k \to \infty$ for a given operator A and a given sequence $\{h_k\}_{k=1}^\infty$ tending to $+\infty$ or to $-\infty$. Typically, these strong limits (if they exist) are simpler than the original operator A, but still keep much information about A. For instance $V_{-h_k} K V_{h_k}$ tends strongly to the zero operator for every compact operator K on the standard Lebesgue space and $V_{-h_k}(aP + Q)V_{h_k}$ tends strongly to $a_l P + Q$ for $h_k \to -\infty$ and to $a_r P + Q$ for $h_k \to +\infty$. For a detailed discussion of the method of limit operators, we refer to the monograph by Rabinovich, Roch, and Silbermann [33].

On variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R})$ the translation operator V_h is, in general, unbounded. So the method of the proof of Theorem 1.2 presented in [6, Section 18.4] should be adjusted accordingly. To this end, we combine ideas from [6, Section 18.4] and [34] (see also [35]). A key lemma concerns the behavior of the sequence $\|V_{h_k} w_k\|_{p(\cdot)}$, where the functions w_k are nice (continuous and decaying faster than $|x|$ as $|x| \to +\infty$): if w_k converges to w and $p(h_k)$ converges to $q \in (1, \infty)$, then $\|V_{h_k} w_k\|_{p(\cdot)}$ converges to the norm of w on the standard Lebesgue space $L^q(\mathbb{R})$. This fact was proved by Rabinovich and Samko [34, Proposition 6.3] for exponents having finite limits at infinity; we relax this hypothesis and assume only that $p \in SO$.

The paper is organized as follows. Section 2 contains the proof of Lemma 1.1. In Section 3 we collect auxiliary material on Fredholmness, injectivity and surjectivity moduli and their relation with invertibility, some fundamental properties of variable Lebesgue spaces. Further, we prove that P and Q are projections on variable Lebesgue spaces and calculate the adjoint operator of $aP + bQ$ with $a, b \in L^\infty_{N\times N}(\mathbb{R})$. We prove that the sequence $\Psi_n = K\chi_{\mathbb{R}\setminus[-n,n]} I$ converges uniformly whenever K is compact on $L^{p(\cdot)}_N(\mathbb{R})$. We finish this section with a property of slowly oscillating functions and an implicit sequence lemma. Both statements play an important role in the proof of the key lemma given in Section 4.

The final Section 5 is devoted to the proof of Theorem 1.2. Let us briefly outline its main steps. First we approximate the operator $A = aP + Q$ by the operators $A_j = a_j P + Q$ where a_j has the same form as a, but with polynomial almost-periodic representatives $a_l^{(j)}$ and $a_l^{(j)}$ at $-\infty$ and $+\infty$, respectively. Since the norm of $K\Psi_n$ is small whenever n is large, from the Fredholmness of A we arrive at an a priori estimate

$$\|\Psi_n f\|_{L^{p(\cdot)}_N(\mathbb{R})} \le \text{const}\|A_j \Psi_n f\|_{L^{p(\cdot)}_N(\mathbb{R})} \quad \text{for} \quad f \in L^{p(\cdot)}_N(\mathbb{R}) \tag{1.2}$$

and large fixed j, n. By the corollary of Kronecker's theorem there exists a sequence $h_m \to +\infty$ such that

$$\|a_r^{(j)}(\cdot + h_m) - a_r^{(j)}(\cdot)\|_{L^\infty_{N\times N}(\mathbb{R})} \to 0 \quad \text{as} \quad m \to \infty. \tag{1.3}$$

If φ is smooth and compactly supported, $\varphi \in [C_c^\infty(\mathbb{R})]_N$, then $\Psi_n V_{h_m}\varphi = V_{h_m}\varphi$ for large m. Hence (1.2) implies that

$$\|V_{h_m}\varphi\|_{L^{p(\cdot)}_N(\mathbb{R})} \le \text{const}\|V_{h_m}(V_{-h_m} A_j V_{h_m}\varphi)\|_{L^{p(\cdot)}_N(\mathbb{R})} \quad \text{for} \quad \varphi \in [C_c^\infty(\mathbb{R})]_N. \tag{1.4}$$

Since the sequence $\{p(h_m)\}$ is bounded, we can extract its subsequence $\{p(h_{m_k})\}$ that converges to a certain number q_r. Taking into account (1.3), we show that the sequence $w_k = V_{-h_{m_k}} A_j V_{h_{m_k}}\varphi$ and the function $w := (a_r^{(j)} P + Q)\varphi$ satisfy the hypotheses of the key lemma. Passing to the limit in (1.4) along the subsequence $\{h_{m_k}\}$ as $k \to \infty$, and then replacing $a_r^{(j)}$ by a_r, we arrive at

$$\|\varphi\|_{L^{q_r}_N(\mathbb{R})} \le \text{const}\|(a_r P + Q)\varphi\|_{L^{q_r}_N(\mathbb{R})} \quad \text{for} \quad \varphi \in [C_c^\infty(\mathbb{R})]_N. \tag{1.5}$$

Applying duality arguments, we also obtain an a priori estimate for the adjoint operator:

$$\|\varphi\|_{L_N^{q_r'}(\mathbb{R})} \le \text{const}\|(a_r P + Q)^* \varphi\|_{L_N^{q_r'}(\mathbb{R})} \quad \text{for} \quad \varphi \in [C_c^\infty(\mathbb{R})]_N \tag{1.6}$$

where $q_r' = q_r/(q_r - 1)$. Since $C_c^\infty(\mathbb{R})$ is dense both in $L^{p(\cdot)}(\mathbb{R})$ and in its dual space $L^{p'(\cdot)}(\mathbb{R})$ whenever (1.1) is fulfilled, from (1.5)–(1.6) it follows that the operator $a_r P + Q$ is invertible on $L_N^{q_r}(\mathbb{R})$.

2. Nontriviality of the class $\mathcal{E}$

2.1. The Hardy-Littlewood maximal operator and the Cauchy singular integral operator

Given $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, the Hardy-Littlewood maximal operator M is defined by

$$(Mf)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy$$

where the supremum is taken over all intervals $Q \subset \mathbb{R}$ containing x. From [11, Theorem 4.8] (see also [20, Theorem 2.7]) and [10, Theorem 8.1] one can extract the following.

Theorem 2.1. *Let $p : \mathbb{R} \to [1, \infty]$ be a measurable function satisfying* (1.1). *If the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R})$, then the Cauchy singular integral operator S is bounded on $L^{p(\cdot)}(\mathbb{R})$.*

Note that in the majority of papers dealing with the boundedness of the Hardy-Littlewood maximal operator it is supposed that the exponent has a finite limit at infinity (see, e.g., [7–9, 16, 23] and the references therein). We refer also to [28, 29], where this condition was weakened and to the recent monograph [12] for the detailed treatment of these questions.

2.2. Lerner's example

One interesting class of variable exponents such that M is bounded on $L^{p(\cdot)}(\mathbb{R})$ was considered by Lerner [27]. Among other things he proved the following.

Theorem 2.2 (Lerner). *There exists an $\alpha > 2$ such that the Hardy-Littlewood maximal operator M is bounded on the variable Lebesgue space $L^{p_L(\cdot)}(\mathbb{R})$ with*

$$p_L(x) := \alpha + \sin\big(\log(\log|x|)\chi_{\{x \in \mathbb{R}: |x| \ge e\}}(x)\big) \quad (x \in \mathbb{R}).$$

Lemma 2.3. *The exponent p_L satisfies* (1.1) *and belongs to $SO \setminus C(\dot{\mathbb{R}})$.*

Proof. It is clear that $p_L \in C(\mathbb{R})$ and p_L is even. Moreover,

$$\lim_{x \to +\infty} x \frac{dp_L(x)}{dx} = \lim_{x \to +\infty} \frac{\cos(\log(\log x))}{\log x} = 0.$$

Then (see, e.g., [2, p. 154–155 and p. 158]) $p_L \in SO$. Obviously,

$$\liminf_{x \to +\infty} p_L(x) = \inf_{x \in \mathbb{R}} p_L(x) = \alpha - 1 > 1, \quad \limsup_{x \to +\infty} p_L(x) = \sup_{x \in \mathbb{R}} p_L(x) = \alpha + 1 < \infty.$$

Thus p_L satisfies (1.1) and $p_L \notin C(\dot{\mathbb{R}})$. □

Lemma 1.1 follows from Theorems 2.1–2.2 and Lemma 2.3.

3. Auxiliary results

3.1. Fredholmness

Recall the following well known fact, which follows from Atkinson's theorem (see, e.g., [14, Chap. 4, Theorem 6.1]).

Lemma 3.1. *Let X be a Banach space and $A, B \in \mathcal{B}(X)$. If A is Fredholm on X and B is invertible on X, then AB and BA are Fredholm on X.*

The next statement is about Fredholmness of adjoints.

Theorem 3.2 (see, e.g., [14, Section 4.15]). *Let X be a Banach space and $A \in \mathcal{B}(X)$. Then A is Fredholm on X if and only if its adjoint A^* is Fredholm on the dual space X^*.*

Let $A \in \mathcal{B}(X)$. An operator $R \in \mathcal{B}(X)$ is said to be a left (resp. right) regularizer of A if $RA - I \in \mathcal{K}(X)$ (resp. $AR - I \in \mathcal{K}(X)$). If R is a left and right regularizer of A, then we say that R is a two-sided regularizer of A.

Theorem 3.3 (see, e.g., [14, Chap. 4, Theorem 7.1]). *Let X be a Banach space. An operator $A \in \mathcal{B}(X)$ is Fredholm on X if and only if there exists a two-sided regularizer of A.*

3.2. Injection and surjection moduli

Let $A \in \mathcal{B}(X)$. Following [30, Sections B.3.1 and B.3.4], consider its injection modulus

$$\mathcal{J}(A; X) := \sup \left\{ c \geq 0 : \ \|Af\|_X \geq c\|f\|_X \text{ for all } f \in X \right\}$$

and its surjection modulus

$$\mathcal{Q}(A; X) := \sup \left\{ c \geq 0 : \ cB_X \subset AB_X \right\}$$

where B_X is the closed unit ball of X. Sometimes these characteristics are also called lower norms of A (see, e.g., [26, Section 1.3]). Fundamental properties of the injection and surjection moduli are collected in the following statements.

Lemma 3.4 (see, e.g., [30, Section B.3.8]). *If $A \in \mathcal{B}(X)$, then*

$$\mathcal{J}(A; X) = \mathcal{Q}(A^*; X^*), \quad \mathcal{Q}(A; X) = \mathcal{J}(A^*; X^*).$$

Lemma 3.5 (see, e.g., [26, Proposition 1.3.7]). *If $A, B \in \mathcal{B}(X)$, then*

$$\mathcal{J}(A; X) \cdot \mathcal{J}(B; X) \leq \mathcal{J}(AB; X), \quad \mathcal{Q}(A; X) \cdot \mathcal{Q}(B; X) \leq \mathcal{Q}(AB; X).$$

Theorem 3.6 (see, e.g., [26, Theorem 1.3.2]). *An operator $A \in \mathcal{B}(X)$ is invertible if and only if*

$$\mathcal{J}(A; X) > 0, \quad \mathcal{Q}(A; X) > 0.$$

If A is invertible, then

$$\mathcal{J}(A; X) = \mathcal{Q}(A; X) = \frac{1}{\|A^{-1}\|_{\mathcal{B}(X)}}.$$

3.3. Some fundamental properties of variable Lebesgue spaces

Let $C_c^\infty(\mathbb{R})$ be the set of all infinitely differentiable functions with compact support. The following results were proved in [24, Theorems 2.4, 2.6, and 2.11].

Theorem 3.7. *Let $p : \mathbb{R} \to [1, \infty]$ be a measurable function satisfying* (1.1) *and $f_n \in L^{p(\cdot)}(\mathbb{R})$. Then*

(a) *the set $C_c^\infty(\mathbb{R})$ is dense in $L^{p(\cdot)}(\mathbb{R})$;*

(b) $\lim\limits_{n\to\infty} I_{p(\cdot)}(f_n) = 0$ *if and only if* $\lim\limits_{n\to\infty} \|f_n\|_{p(\cdot)} = 0$*;*

(c) *for every continuous linear functional G on $L^{p(\cdot)}(\mathbb{R})$ there exists a unique function $g \in L^{p'(\cdot)}(\mathbb{R})$ such that*

$$G(f) = \int_{\mathbb{R}} f(x)\overline{g(x)}dx \quad \text{for } f \in L^{p(\cdot)}(\mathbb{R})$$

and the norms $\|G\|$ and $\|g\|_{p'(\cdot)}$ are equivalent.

Corollary 3.8. *Let $p : \mathbb{R} \to [1, \infty]$ be a measurable function satisfying* (1.1)*. For every continuous linear functional G on $L_N^{p(\cdot)}(\mathbb{R})$ there exists a unique function $g = (g_1, \dots, g_N) \in L_N^{p'(\cdot)}(\mathbb{R})$ such that*

$$G(f) = \sum_{\alpha=1}^{N} \int_{\mathbb{R}} f_\alpha(x)\overline{g_\alpha(x)}dx =: \langle f, g\rangle \tag{3.1}$$

for all $f = (f_1, \dots, f_N) \in L_N^{p(\cdot)}(\mathbb{R})$. The norms of $\|G\|$ and $\|g\|_{L_N^{p'(\cdot)}(\mathbb{R})}$ are equivalent.

3.4. Singular integral operators and their adjoints

For $a \in L^\infty_{N\times N}(\mathbb{R})$, let a^* denote the complex conjugate of the transpose matrix function a^{T}.

Lemma 3.9. *Let $p : \mathbb{R} \to [1, \infty]$ be a measurable function satisfying* (1.1)*. If $a \in L^\infty_{N\times N}(\mathbb{R})$, then*

$$(aI)^* = a^*I \in \mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R})).$$

Proof. Let $\langle\cdot,\cdot\rangle$ be the pairing defined by (3.1) and $f \in L_N^{p(\cdot)}(\mathbb{R})$, $g \in L_N^{p'(\cdot)}(\mathbb{R})$. Then

$$\begin{aligned}
\langle af, g\rangle &= \sum_{\alpha=1}^{N} \int_{\mathbb{R}} \left(\sum_{\beta=1}^{N} a_{\alpha\beta}(x) f_\beta(x)\right) \overline{g_\alpha(x)}\, dx = \sum_{\beta=1}^{N} \int_{\mathbb{R}} \left(\sum_{\alpha=1}^{N} a_{\alpha\beta}(x)\overline{g_\alpha(x)}\right) f_\beta(x)\, dx \\
&= \sum_{\alpha=1}^{N} \int_{\mathbb{R}} \left(\sum_{\beta=1}^{N} a_{\beta\alpha}(x)\overline{g_\beta(x)}\right) f_\alpha(x)\, dx = \sum_{\alpha=1}^{N} \int_{\mathbb{R}} f_\alpha(x) \overline{\left(\sum_{\beta=1}^{N} \overline{a_{\beta\alpha}(x)} g_\beta(x)\right)}\, dx = \langle f, a^*g\rangle,
\end{aligned}$$

which completes the proof in view of Corollary 3.8. $\square$

Lemma 3.10. *If $p \in \mathcal{E}$, then $P, Q \in \mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))$ and $P^2 = P$, $Q^2 = Q$.*

Proof. Since the operators S, P, and Q are defined elementwise, it is sufficient to prove the statement for $N = 1$. It is well known (see, e.g., [13, formula (3.5)]) that

$$S^2\varphi = \varphi \quad \text{for} \quad \varphi \in L^2(\mathbb{R}).$$

In particular, the above formula holds for all $\varphi \in C_c^\infty(\mathbb{R})$. Let $f \in L^{p(\cdot)}(\mathbb{R})$. By Theorem 3.7(a), there exists a sequence $\{\varphi_n\}_{n=1}^\infty \subset C_c^\infty(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|f - \varphi_n\|_{p(\cdot)} = 0. \tag{3.2}$$

Since $p \in \mathcal{E}$, we conclude that $S^2 \in \mathcal{B}(L^{p(\cdot)}(\mathbb{R}))$. Hence

$$\lim_{n\to\infty} \|S^2 f - S^2\varphi_n\|_{p(\cdot)} \le \|S^2\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} \lim_{n\to\infty} \|f - \varphi_n\|_{p(\cdot)} = 0. \tag{3.3}$$

Passing to the limit in the equality $S^2\varphi_n = \varphi_n$ as $n \to \infty$ and taking into account (3.2)–(3.3), we arrive at $S^2 f = f$ for $f \in L^{p(\cdot)}(\mathbb{R})$, that is, $S^2 = I$ on $L^{p(\cdot)}(\mathbb{R})$. This immediately implies that $P^2 = P$ and $Q^2 = Q$. □

Lemma 3.11. *If $p \in \mathcal{E}$, then $p' \in \mathcal{E}$ and*

$$S^* = S, \quad P^* = P, \quad Q^* = Q$$

belong to $\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))$.

Proof. Since the operators S, P, and Q are defined elementwise on $L^{p(\cdot)}(\mathbb{R})$, it is sufficient to prove the statement for $N = 1$. It is well known that for $\varphi, \psi \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} (S\varphi)(x)\overline{\psi(x)}\,dx = \int_{\mathbb{R}} \varphi(x)\overline{(S\psi)(x)}\,dx$$

(see, e.g., [13, formula (3.6)]). In particular, this equality holds for all $\varphi, \psi \in C_c^\infty(\mathbb{R})$. This means that S is a self-adjoint and densely defined operator on $L^{p(\cdot)}(\mathbb{R})$ and $L^{p'(\cdot)}(\mathbb{R})$ (see Theorem 3.7(a)). By the standard argument (see [22, Chap. III, Section 5.5]), one can show that $S = S^* \in \mathcal{B}(L^{p'(\cdot)}(\mathbb{R}))$ because $S \in \mathcal{B}(L^{p(\cdot)}(\mathbb{R}))$. This yields $p' \in \mathcal{E}$ and also the equalities

$$P^* = (I + S)^*/2 = (I + S)/2 = P, \quad Q^* = (I - S)^*/2 = (I - S)/2 = Q,$$

which finishes the proof. □

From Lemmas 3.9 and 3.11 we immediately get the following.

Corollary 3.12. *If $p \in \mathcal{E}$ and $a, b \in L_{N\times N}^\infty(\mathbb{R})$, then*

$$(aP + bQ)^* = Pa^*I + Qb^*I \in \mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R})).$$

The proof of the next statement is a matter of a straightforward calculation and application of Lemma 3.10 when necessary.

Lemma 3.13. *If $p \in \mathcal{E}$ and $a \in L_{N\times N}^\infty(\mathbb{R})$, then*

$$(I \pm PaQ)^{-1} = I \mp PaQ, \quad (I \pm QaP)^{-1} = I \mp QaP, \tag{3.4}$$

and

$$PaI + Q = (I + PaQ)(aP + Q)(I - QaP), \quad P + QaI = (I + QaP)(P + aQ)(I - PaQ).$$

3.5. Compact operators and convergence of sequences of operators

Lemma 3.14 (see, e.g., [36, Lemma 1.4.7]). *Let X be a Banach space. Suppose $A, B \in \mathcal{B}(X)$, and $A_n, B_n \in \mathcal{B}(X)$ for all $n \in \mathbb{N}$. If $K \in \mathcal{K}(X)$ and if $A_n \to A$ and $B_n^* \to B^*$ strongly as $n \to \infty$, then $\|A_n K B_n - AKB\|_{\mathcal{B}(X)} \to 0$ as $n \to \infty$.*

Let χ_E be the characteristic function of a set $E \subset \mathbb{R}$.

Lemma 3.15. *Let $p : \mathbb{R} \to [1, \infty]$ be a measurable function satisfying* (1.1). *For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, put*

$$\psi_n(x) := 1 - \chi_{[-1,1]}(x/n).$$

(a) *The sequence* $\{\psi_n I\}_{n=1}^{\infty}$ *converges strongly to the zero operator on* $L^{p(\cdot)}(\mathbb{R})$ *and on* $L^{p'(\cdot)}(\mathbb{R})$ *as* $n \to \infty$.
(b) *If* $K \in \mathcal{K}(L^{p(\cdot)}(\mathbb{R}))$, *then*

$$\lim_{n\to\infty} \|K\psi_n I\|_{\mathcal{B}(L^{p(\cdot)}(\mathbb{R}))} = 0.$$

Proof. (a) If $1 < \operatorname*{ess\,inf}_{x\in\mathbb{R}} p(x)$, then $\operatorname*{ess\,sup}_{x\in\mathbb{R}} p'(x) < \infty$. Therefore, by Theorem 3.7(a)–(b), it is sufficient to prove that

$$\lim_{n\to\infty} I_{p(\cdot)}(\psi_n f) = 0 \quad \text{for all} \quad f \in C_c^\infty(\mathbb{R}). \tag{3.5}$$

Suppose $f \in C_c^\infty(\mathbb{R})$. Then there exists $n_0 \in \mathbb{N}$ such that $\operatorname{supp} f \subset [-n_0, n_0]$. Then for all $n \geq n_0$,

$$I_{p(\cdot)}(\psi_n f) = \int_{\mathbb{R}} |\big(1 - \chi_{[-1,1]}(x/n)\big) f(x)|^{p(x)} dx = \int_{\mathbb{R}} |\chi_{\mathbb{R}\setminus[-n,n]}(x) f(x)|^{p(x)} dx = \int_{\mathbb{R}\setminus[-n,n]} |f(x)|^{p(x)} dx = 0.$$

Thus $I_{p(\cdot)}(\psi_n f) = 0$ for all $n \geq n_0$, which finishes the proof of (3.5). Part (a) is proved.

(b) From Theorem 3.7(c) it follows that $(\psi_n I)^* = \psi_n I \in \mathcal{B}(L^{p'(\cdot)}(\mathbb{R}))$. By part (a), the sequence $\{(\psi_n I)^*\}_{n=1}^{\infty}$ converges strongly to the zero operator. It remains to apply Lemma 3.14. ☐

3.6. Important property of slowly oscillating functions

The following statement is proved by analogy with [3, Proposition 4(ii)].

Lemma 3.16. *Let* $f \in SO$. *Suppose* $\{h_k\}_{k=1}^{\infty} \subset \mathbb{R}$ *is a sequence tending to* $+\infty$ (*resp. to* $-\infty$) *and such that the limit*

$$\lim_{k\to\infty} f(h_k) =: g \tag{3.6}$$

exist. Then for every $R > 0$,

$$\lim_{k\to\infty} \sup_{x\in[-R,R]} |f(x + h_k) - g| = 0. \tag{3.7}$$

Proof. For every $k \in \mathbb{N}$,

$$\begin{aligned} \sup_{x\in[-R,R]} |f(x+h_k) - g| &\leq \sup_{x\in[-R,R]} |f(x+h_k) - f(h_k)| + |f(h_k) - g| \\ &\leq \sup_{x,y\in[h_k-R,h_k+R]} |f(x) - f(y)| + |f(h_k) - g|. \end{aligned} \tag{3.8}$$

Let for definiteness $\lim_{k\to\infty} h_k = -\infty$. Then there exists a $k_0 \in \mathbb{N}$ such that $h_k \leq -3R$ for all $k \geq k_0$. Therefore $2(h_k + R) \leq h_k - R$ and

$$[2(h_k + R), h_k + R] \supset [h_k - R, h_k + R].$$

Thus for $k \geq k_0$,

$$\begin{aligned} \sup_{x,y\in[h_k-R,h_k+R]} |f(x) - f(y)| &\leq \sup_{x,y\in[2(h_k+R),h_k+R]} |f(x) - f(y)| \\ &\leq \operatorname{osc}\big(f, [2(h_k + R), h_k + R] \cup [-(h_k + R), -2(h_k + R)]\big). \end{aligned} \tag{3.9}$$

Since $f \in SO$, the latter oscillation tends to zero as $k \to \infty$. Combining this observation with (3.6) and (3.8)–(3.9), we arrive at (3.7). ☐

3.7. Lemma on an implicit sequence

We will need the following result from Elementary Calculus. Put $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$.

Lemma 3.17. *Let $F : \mathbb{R}_+ \times (\mathbb{N} \cup \{\infty\}) \to \mathbb{R}_+$ be a function such that*

(i) *for every $k \in \mathbb{N} \cup \{\infty\}$, the function $F(\cdot, k)$ is continuous and strictly decreasing;*
(ii) *for every $\lambda \in \mathbb{R}_+$,*

$$\lim_{k\to\infty} F(\lambda, k) = F(\lambda, \infty). \tag{3.10}$$

If $F(\lambda_\infty, \infty) = 1$ for some $\lambda_\infty \in \mathbb{R}_+$, then there exists a number $k_0 \in \mathbb{N}$ and a unique sequence $\{\lambda(k)\}_{k=k_0}^{\infty}$ such that $F(\lambda(k), k) = 1$ for all $k \geq k_0$ and

$$\lim_{k\to\infty} \lambda(k) = \lambda_\infty. \tag{3.11}$$

Proof. The proof is developed by analogy with the proof of the lemma from [25, Section 41.1].

Let $\varepsilon \in (0, \lambda_\infty/2]$. Since $F(\cdot, \infty)$ is strictly decreasing,

$$F(\lambda_\infty + \varepsilon, \infty) < F(\lambda_\infty, \infty) = 1 < F(\lambda_\infty - \varepsilon, \infty). \tag{3.12}$$

From (3.10) it follows that there exist $k_+(\varepsilon), k_-(\varepsilon) \in \mathbb{N}$ such that

$$|F(\lambda_\infty + \varepsilon, \infty) - F(\lambda_\infty + \varepsilon, k)| < \frac{1 - F(\lambda_\infty + \varepsilon, \infty)}{2} \tag{3.13}$$

for $k \geq k_+(\varepsilon)$ and

$$|F(\lambda_\infty - \varepsilon, \infty) - F(\lambda_\infty - \varepsilon, k)| < \frac{F(\lambda_\infty - \varepsilon, \infty) - 1}{2} \tag{3.14}$$

for $k \geq k_-(\varepsilon)$. Let

$$k_0(\varepsilon) := \max\{k_-(\varepsilon), k_+(\varepsilon)\}, \quad k_0 := k_0(\lambda_\infty/2).$$

Taking into account (3.12), we obtain from (3.13)–(3.14) that

$$\begin{aligned} F(\lambda_\infty + \varepsilon, k) &< \frac{1 - F(\lambda_\infty + \varepsilon, \infty)}{2} + F(\lambda_\infty + \varepsilon, \infty) = \frac{1 + F(\lambda_\infty + \varepsilon, \infty)}{2} < 1, \\ F(\lambda_\infty - \varepsilon, k) &> F(\lambda_\infty - \varepsilon, \infty) - \frac{F(\lambda_\infty - \varepsilon, \infty) - 1}{2} = \frac{1 + F(\lambda_\infty - \varepsilon, \infty)}{2} > 1. \end{aligned}$$

Thus, for all $k \geq k_0(\varepsilon)$,

$$F(\lambda_\infty + \varepsilon, k) < 1 < F(\lambda_\infty - \varepsilon, k). \tag{3.15}$$

Since $F(\cdot, k)$ is continuous in the first variable for every fixed k, from (3.15) we see, by the Bolzano-Cauchy intermediate value theorem, that there exists a $\lambda(k)$ such that $F(\lambda(k), k) = 1$ and

$$\lambda_\infty - \varepsilon < \lambda(k) < \lambda_\infty + \varepsilon. \tag{3.16}$$

The value $\lambda(k)$ is unique for every k because $F(\cdot, k)$ is strictly decreasing. Thus, for every $\varepsilon \in (0, \infty/2]$, there exists a number $k_0(\varepsilon) \in \mathbb{N}$ such that for all $k \geq k_0(\varepsilon)$, inequality (3.16) holds, which implies (3.11). $\square$

4. Norms of translations of decaying continuous functions

4.1. Technical lemma

We start with the following technical statement.

Lemma 4.1. *Suppose $p : \mathbb{R} \to (1,\infty)$ belongs to SO and satisfies (1.1). Let $\{h_k\}_{k=1}^{\infty} \subset \mathbb{R}$ be a sequence tending to $+\infty$ (resp. to $-\infty$) and such that the limit*

$$\lim_{k\to\infty} p(h_k) =: q$$

exists. Suppose $R > 0$ and $\{w_k\}_{k=1}^{\infty} \subset C(\mathbb{R})$ is a sequence which converges pointwise to a function $w \in C(\mathbb{R})$ on the segment $[-R, R]$. If there are positive constants $C_1 < C_2$ and a measurable set $\Delta \subset [-R, R]$ such that for all sufficiently large k and all $x \in [-R, R] \setminus \Delta$,

$$C_1 \le w_k(x) \le C_2, \quad C_1 \le w(x) \le C_2, \tag{4.1}$$

then for every $\lambda \in \mathbb{R}_+$,

$$\lim_{k\to\infty} \int_{[-R,R]\setminus\Delta} \left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} dx = \int_{[-R,R]\setminus\Delta} \left|\frac{w(x)}{\lambda}\right|^{q} dx. \tag{4.2}$$

Proof. The proof is based on the Lebesgue bounded convergence theorem (see, e.g., [1, Theorem 10.29]). Let us show that for all $\lambda \in \mathbb{R}_+$ and all $x \in [-R, R] \setminus \Delta$,

$$\lim_{k\to\infty} \left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} = \left|\frac{w(x)}{\lambda}\right|^{q}. \tag{4.3}$$

By the mean value theorem,

$$\begin{aligned}\left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} - \left|\frac{w(x)}{\lambda}\right|^{q} &= \exp\left(p(x+h_k)\log\left|\frac{w_k(x)}{\lambda}\right|\right) - \exp\left(q\log\left|\frac{w(x)}{\lambda}\right|\right) \\ &= e^{\xi}\left(p(x+h_k)\log\left|\frac{w_k(x)}{\lambda}\right| - q\log\left|\frac{w(x)}{\lambda}\right|\right),\end{aligned}$$

where ξ is some real number between

$$p(x+h_k)\log\left|\frac{w_k(x)}{\lambda}\right| \quad \text{and} \quad q\log\left|\frac{w(x)}{\lambda}\right|.$$

Taking into account that there exists a $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ inequalities (4.1) are fulfilled, we have

$$p(x+h_k)\log\left|\frac{w_k(x)}{\lambda}\right| \le p(x+h_k)\log\frac{C_2}{\lambda} \le p(x+h_k)\left|\log\frac{C_2}{\lambda}\right| \le p_+\left|\log\frac{C_2}{\lambda}\right|$$

and

$$q\log\left|\frac{w(x)}{\lambda}\right| \le q\log\frac{C_2}{\lambda} \le q\left|\log\frac{C_2}{\lambda}\right| \le p_+\left|\log\frac{C_2}{\lambda}\right|.$$

Hence

$$\xi \le p_+\left|\log\frac{C_2}{\lambda}\right| \quad \text{and} \quad e^{\xi} \le \exp\left(p_+\left|\log\frac{C_2}{\lambda}\right|\right) =: C_3.$$

Then for all $k \ge k_0$,

$$\begin{aligned}\left|\left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} - \left|\frac{w(x)}{\lambda}\right|^{q}\right| \le& C_3\left|p(x+h_k)\log\left|\frac{w_k(x)}{\lambda}\right| - q\log\left|\frac{w(x)}{\lambda}\right|\right| \\ \le& C_3|p(x+h_k) - q|\,\left|\log\left|\frac{w_k(x)}{\lambda}\right|\right| + C_3 q\left|\log\left|\frac{w_k(x)}{\lambda}\right| - \log\left|\frac{w(x)}{\lambda}\right|\right|. \end{aligned} \tag{4.4}$$

Further, we have

$$\log\left|\frac{w_k(x)}{\lambda}\right| \le \log\frac{C_2}{\lambda} \le \left|\log\frac{C_2}{\lambda}\right| \le \max\left\{\left|\log\frac{C_1}{\lambda}\right|, \left|\log\frac{C_2}{\lambda}\right|\right\},$$
$$\log\left|\frac{w_k(x)}{\lambda}\right| \ge \log\frac{C_1}{\lambda} \ge -\left|\log\frac{C_1}{\lambda}\right| \ge -\max\left\{\left|\log\frac{C_1}{\lambda}\right|, \left|\log\frac{C_2}{\lambda}\right|\right\}.$$

Therefore, for all $k \ge k_0$,

$$\left|\log\left|\frac{w_k(x)}{\lambda}\right|\right| \le \max\left\{\left|\log\frac{C_1}{\lambda}\right|, \left|\log\frac{C_2}{\lambda}\right|\right\} =: C_4 < \infty. \tag{4.5}$$

Applying the main value theorem once again, we see that

$$\log\left|\frac{w_k(x)}{\lambda}\right| - \log\left|\frac{w(x)}{\lambda}\right| = \frac{1}{\zeta}\big(|w_k(x)| - |w(x)|\big),$$

where ζ is some number between $|w_k(x)|$ and $|w(x)|$. Hence $\zeta \in [C_1, C_2]$. Then for all $k \ge k_0$,

$$\left|\log\left|\frac{w_k(x)}{\lambda}\right| - \log\left|\frac{w(x)}{\lambda}\right|\right| \le \frac{1}{C_1}\big||w_k(x)| - |w(x)|\big| \le \frac{1}{C_1}|w_k(x) - w(x)|. \tag{4.6}$$

Combining (4.4)–(4.6), we arrive at

$$\left|\left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} - \left|\frac{w(x)}{\lambda}\right|^{q}\right| \le C_3C_4|p(x+h_k) - q| + \frac{C_3 q}{C_1}|w_k(x) - w(x)| \tag{4.7}$$

for all $k \ge k_0$. From Lemma 3.16 it follows that

$$\lim_{k\to\infty} |p(x+h_k) - q| = 0. \tag{4.8}$$

But it is given that

$$\lim_{k\to\infty} |w_k(x) - w(x)| = 0. \tag{4.9}$$

Thus, from inequality (4.7) and equalities (4.8)–(4.9) we immediately get (4.3).

Further, for every $x \in [-R, R] \setminus \Delta$ and $k \ge k_0$,

$$\left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} \le \left(\frac{C_2}{\lambda}\right)^{p(x+h_k)} \le \left(\max\left\{1, \frac{C_2}{\lambda}\right\}\right)^{p(x+h_k)} \le \left(\max\left\{1, \frac{C_2}{\lambda}\right\}\right)^{p_+}$$

because $p(x+h_k) \le p_+$. Thus, the sequence $|w_k(x)/\lambda|^{p(x+h_k)}$ is uniformly bounded and converges pointwise to $|w(x)/\lambda|^q$. By the Lebesgue bounded convergence theorem, this yields (4.2). □

4.2. Key lemma

The key to the proof of Theorem 1.2 is the following generalization of the one-dimensional version of [34, Proposition 6.3]. Note that conditions on p imposed in [34] imply that $p \in C(\dot{\mathbb{R}})$. For the readers' convenience, we provide here a detailed proof in our more general situation, though the outline remains more or less the same as in [34].

Lemma 4.2. *Suppose $p : \mathbb{R} \to (1, \infty)$ belongs to SO and satisfies (1.1). Let $\{h_k\}_{k=1}^{\infty} \subset \mathbb{R}$ be a sequence tending to $+\infty$ (resp. to $-\infty$) and such that the limit*

$$\lim_{k\to\infty} p(h_k) =: q$$

exists. Suppose $w \in C(\dot{\mathbb{R}})$ and $\{w_k\}_{k=1}^{\infty} \subset C(\dot{\mathbb{R}})$ are such that

(i) *for all* $x \in \mathbb{R}$,

$$\lim_{k\to\infty} w_k(x) = w(x),$$

and this convergence is uniform on each closed segment $J \subset \mathbb{R}_+$*;*

(ii) *there exists a constant* $C > 0$ *such that for all* $k \in \mathbb{N}$ *and* $x \in \mathbb{R}$,

$$|w(x)| \le \frac{C}{1+|x|}, \quad |w_k(x)| \le \frac{C}{1+|x|}.$$

Then

$$\lim_{k\to\infty} \|V_{h_k} w_k\|_{p(\cdot)} = \|w\|_q. \tag{4.10}$$

Proof. For $\lambda > 0$ and $k \in \mathbb{N}$, put

$$F(\lambda,k) := \int_{\mathbb{R}} \left|\frac{(V_{h_k}w_k)(x)}{\lambda}\right|^{p(x)} dx = \int_{\mathbb{R}} \left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} dx, \quad F(\lambda,\infty) := \int_{\mathbb{R}} \left|\frac{w(x)}{\lambda}\right|^{q} dx = \lambda^{-q}\|w\|_q^q.$$

First, let us show that for every $\lambda > 0$,

$$\lim_{k\to\infty} F(\lambda,k) = F(\lambda,\infty). \tag{4.11}$$

Fix some numbers $R > 0$ and $\delta > 0$. We will specify the choice of R and δ later. Consider the (possibly empty) set

$$\Delta_\delta := \big\{x \in [-R,R] : \ |w(x)| \le 2\delta\big\} \tag{4.12}$$

and put

$$T_R(\lambda,k) := \int_{|x|>R} \left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} dx, \qquad T_R(\lambda,\infty) := \int_{|x|>R} \left|\frac{w(x)}{\lambda}\right|^{q} dx,$$

$$L_{\delta,R}(\lambda,k) := \int_{\Delta_\delta} \left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} dx, \qquad L_{\delta,R}(\lambda,\infty) := \int_{\Delta_\delta} \left|\frac{w(x)}{\lambda}\right|^{q} dx,$$

and

$$D_{\delta,R}(\lambda,k) := \left|\int_{[-R,R]\setminus\Delta_\delta} \left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} dx - \int_{[-R,R]\setminus\Delta_\delta} \left|\frac{w(x)}{\lambda}\right|^{q} dx\right|.$$

Here "T" is for "tail", "L" is for "little", and "D" is for "difference". It is clear that

$$|F(\lambda,k) - F(\lambda,\infty)| \le T_R(\lambda,k) + T_R(\lambda,\infty) + L_{\delta,R}(\lambda,k) + L_{\delta,R}(\lambda,\infty) + D_{\delta,R}(\lambda,k). \tag{4.13}$$

Fix $\varepsilon > 0$. First we will show that it is possible to choose R so large that for $k \in \mathbb{N}$,

$$T_R(\lambda,k) + T_R(\lambda,\infty) < \varepsilon/3. \tag{4.14}$$

Let for the moment $R \ge C/\lambda$. Then from (1.1) and hypothesis (ii) we obtain

$$\left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} \le \left(\frac{C}{\lambda|x|}\right)^{p(x+h_k)} \le \left(\frac{C}{\lambda|x|}\right)^{p_-} \quad \text{for} \quad |x| \ge R.$$

Then for $\lambda > 0$, $k \in \mathbb{N}$, and $R \ge C/\lambda$,

$$T_R(\lambda,k) \le \int_{|x|>R} \left(\frac{C}{\lambda|x|}\right)^{p_-} dx = 2\left(\frac{C}{\lambda}\right)^{p_-} \int_R^{+\infty} \frac{dx}{x^{p_-}} = \frac{2}{p_- - 1}\left(\frac{C}{\lambda}\right)^{p_-} R^{1-p_-} \tag{4.15}$$

and analogously

$$T_R(\lambda,\infty) \le \frac{2}{p_- - 1}\left(\frac{C}{\lambda}\right)^{p_-} R^{1-p_-} \tag{4.16}$$

(recall that $q \geq p_- > 1$). We choose R as the solution of the equation

$$\frac{4}{p_- - 1}\left(\frac{C}{\lambda}\right)^{p_-} R^{1-p_-} = \frac{\varepsilon}{6}. \tag{4.17}$$

Then from inequalities (4.15)–(4.16) it follows that inequality (4.14) holds.

It remains to show that for so chosen R one has $R \geq C/\lambda$ whenever ε is sufficiently small. Indeed, from (4.17) we obtain

$$R = \left(\frac{24}{p_- - 1}\right)^{1/(p_- - 1)} \left(\frac{C}{\lambda}\right)^{p_-/(p_- - 1)} \left(\frac{1}{\varepsilon}\right)^{1/(p_- - 1)}, \tag{4.18}$$

and $R \geq C/\lambda$ is equivalent to

$$\left(\frac{24}{p_- - 1}\right)^{1/(p_- - 1)} \frac{C}{\lambda} \geq \varepsilon^{1/(p_- - 1)}.$$

That is, if

$$0 < \varepsilon \leq \frac{24}{p_- - 1}\left(\frac{C}{\lambda}\right)^{p_- - 1} =: \varepsilon_1,$$

then R given by (4.18) satisfies $R \geq C/\lambda$ and inequality (4.14) holds.

Now we will choose $\delta > 0$ and $k_0 \in \mathbb{N}$ such that for $k \geq k_0$,

$$L_{\delta,R}(\lambda, k) + L_{\delta,R}(\lambda, \infty) < \varepsilon/3. \tag{4.19}$$

Let for the moment δ is so that $3\delta/\lambda \leq 1$. For R and δ, by hypothesis (i), there exists a $k_0 := k_0(\varepsilon) = k_0(\delta, R) \in \mathbb{N}$ such that for all $x \in [-R, R]$ and all $k \geq k_0$,

$$|w_k(x) - w(x)| < \delta.$$

Hence, for all $k \geq k_0$,

$$|w(x)| - \delta \leq |w_k(x)| \leq |w(x)| + \delta. \tag{4.20}$$

From (4.12) and (4.20) we see that for $k \geq k_0$ and $x \in \Delta_\delta$,

$$\left|\frac{w(x)}{\lambda}\right| \leq \frac{2\delta}{\lambda}, \quad \left|\frac{w_k(x)}{\lambda}\right| \leq \frac{3\delta}{\lambda}.$$

Hence, taking into account that $p(x + h_k) > 1$ and $q > 1$, we have for $k \geq k_0$,

$$L_{\delta,R}(\lambda, k) \leq \int_{\Delta_\delta} \left(\frac{3\delta}{\lambda}\right)^{p(x+h_k)} dx \leq \frac{3\delta}{\lambda}\int_{\Delta_\delta} dx \leq \frac{6\delta R}{\lambda}, \tag{4.21}$$

$$L_{\delta,R}(\lambda, \infty) \leq \int_{\Delta_\delta} \left(\frac{2\delta}{\lambda}\right)^{q} dx \leq \frac{2\delta}{\lambda}\int_{\Delta_\delta} dx \leq \frac{4\delta R}{\lambda}. \tag{4.22}$$

Let us choose δ as the solution of the equation

$$\frac{10\delta R}{\lambda} = \frac{\varepsilon}{6}. \tag{4.23}$$

Then from inequalities (4.21)–(4.22) it follows that inequality (4.19) is fulfilled for all $k \geq k_0$.

It remains to show that we can guarantee that $3\delta/\lambda \leq 1$ whenever ε is sufficiently small. Indeed, from (4.18) and (4.23) we see that

$$\frac{3\delta}{\lambda} = \frac{\varepsilon}{20R} = \frac{\varepsilon}{20}\left(\frac{p_- - 1}{24}\right)^{1/(p_- - 1)} \left(\frac{\lambda}{C}\right)^{p_-/(p_- - 1)} \varepsilon^{1/(p_- - 1)} \leq 1$$

is equivalent to

$$\varepsilon^{2/(p_--1)} \le 20 \left(\frac{24}{p_- - 1}\right)^{1/(p_--1)} \left(\frac{C}{\lambda}\right)^{p_-/(p_--1)}.$$

That is, if

$$0 < \varepsilon \le 20^{(p_--1)/2} \left(\frac{24}{p_- - 1}\right)^{1/2} \left(\frac{C}{\lambda}\right)^{p_-/2} =: \varepsilon_2,$$

then $3\delta/\lambda \le 1$. Thus, if $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2\})$, then we can choose $R > 0$ by (4.18), $\delta > 0$ as the solution of (4.23), and then choose a $k_0 = k_0(\delta, R)$ such that for all $k \ge k_0$, inequalities (4.14) and (4.19) are fulfilled. From (4.13), (4.14), and (4.19) we get

$$|F(\lambda, k) - F(\lambda, \infty)| \le 2\varepsilon/3 + D_{\delta,R}(\lambda, k) \quad \text{for} \quad k \ge k_0. \tag{4.24}$$

From (4.12) and (4.20) it follows that for $x \in [-R, R] \setminus \Delta_\delta$ and $k \ge k_0$,

$$2\delta < |w(x)| \le C, \quad \delta < |w(x)| \le C.$$

From Lemma 4.1 we deduce that there exists $k_1(\varepsilon) \ge k_0$ such that

$$D_{\delta,R}(\lambda, k) < \varepsilon/3 \quad \text{for} \quad k \ge k_1(\varepsilon). \tag{4.25}$$

Combining (4.24) and (4.25), we see that for $\varepsilon > 0$ there exists a $k_1(\varepsilon) \in \mathbb{N}$ such that for all $k \ge k_1(\varepsilon)$,

$$|F(\lambda, k) - F(\lambda, \infty)| < \varepsilon,$$

which finishes the proof of (4.11).

If the limit function w is equal to zero identically on $\mathbb{R}$, then from equality (4.11) we have

$$\lim_{k\to\infty} I_{p(\cdot)}(V_{h_k} w_k) = 0.$$

Then from Theorem 3.7(b) we obtain that

$$\lim_{k\to\infty} \|V_{h_k} w_k\|_{p(\cdot)} = 0 = \|w\|_q,$$

which finishes the proof of the lemma in the case $\|w\|_q = 0$.

Assume now that $\|w\|_q > 0$. Then, obviously, the function $F(\lambda, \infty) = \lambda^{-q}\|w\|_q$ is strictly decreasing and continuous in $\lambda \in \mathbb{R}$. Moreover,

$$F(\|w\|_q, \infty) = 1. \tag{4.26}$$

Without loss of generality we may assume that all functions w_k are not identically zero on $\mathbb{R}$. Let us show that for each $k \in \mathbb{N}$, the function $F(\lambda, k)$ is strictly decreasing and continuous with respect to $\lambda \in \mathbb{R}_+$. Clearly, for every $k \in \mathbb{N}$, $x \in \mathbb{R}$, and $\lambda \in \mathbb{R}_+$,

$$\frac{\partial}{\partial\lambda}\left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} = -\frac{p(x+h_k)}{\lambda}\left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)}.$$

Let $[\alpha, \beta] \subset \mathbb{R}_+$ be some segment. It is not difficult to see that for all $\lambda \in [\alpha, \beta]$,

$$\left|\frac{\partial}{\partial\lambda}\left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)}\right| \le \frac{p_+}{\alpha}\left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} = \frac{p_+}{\alpha} F(\alpha, k) < \infty.$$

Therefore, by the theorem on the differentiation under the sign of the Lebesgue integral (see, e.g., [1, Theorem 10.39]), the function $F(\lambda, k)$ is differentiable in $\lambda \in (\alpha, \beta)$ and

$$\frac{\partial F}{\partial\lambda}(\lambda, k) = -\int_{\mathbb{R}} \frac{p(x+h_k)}{\lambda}\left|\frac{w_k(x)}{\lambda}\right|^{p(x+h_k)} dx.$$

Since $[\alpha,\beta]$ was chosen arbitrarily, we conclude that $F(\lambda,k)$ is differentiable in the first variable on $\mathbb{R}_+$ and

$$\frac{\partial F}{\partial\lambda}(\lambda,k)<0 \quad \text{for} \quad \lambda\in\mathbb{R}_+.$$

Thus, $F(\lambda,k)$ is strictly increasing and continuous in $\lambda\in\mathbb{R}_+$. From this observation, (4.11), and (4.26) we obtain in view of Lemma 3.17 that there exist a number $k_2\in\mathbb{N}$ and a unique sequence $\{\lambda(k)\}_{k=k_2}^{\infty}$ such that $F(\lambda(k),k)=1$ for all $k\ge k_2$ and

$$\lim_{k\to\infty}\lambda(k)=\|w\|_q. \tag{4.27}$$

On the other hand, taking into account that $F(\lambda,k)$ is strictly decreasing and continuous, we see that

$$\|V_{h_k}w_k\|_{p(\cdot)}=\inf\big\{\lambda>0:\ F(\lambda,k)\le 1\big\}=\lambda(k). \tag{4.28}$$

Combining (4.27) and (4.28), we arrive at (4.10). □

5. Proof of the main result

5.1. Verification of the hypotheses of the key lemma

We start with the following consequence of the Kronecker theorem on almost periodic functions (see, e.g., [6, Theorem 1.12]).

Lemma 5.1 (see [6, Lemma 10.2]). *If $a_1,\dots,a_M\in AP^0_{N\times N}$ is a finite collection of almost periodic polynomials, then there exists a sequence $\{h_m\}_{m=1}^{\infty}\subset\mathbb{R}$ such that $h_m\to+\infty$ (resp. $h_m\to-\infty$) as $m\to\infty$ and*

$$\lim_{m\to\infty}\|a_j(\cdot+h_m)-a_j(\cdot)\|_{L^\infty_{N\times N}(\mathbb{R})}=0$$

for all $j\in\{1,\dots,M\}$.

The operator S behaves extremely well on smooth compactly supported functions. More precisely, we have the following.

Lemma 5.2. *If $\varphi\in C_c^\infty(\mathbb{R})$, then $S\varphi\in C(\mathbb{R})$ and there is a constant $C_\varphi>0$ such that*

$$|(S\varphi)(x)|\le\frac{C_\varphi}{1+|x|}\quad (x\in\mathbb{R}).$$

Proof. The continuity of $S\varphi$ is a consequence of the Privalov theorem (see, e.g., [38, Chap. II, Section 6.9]). For the pointwise estimate for $S\varphi$, see e.g. [15, Exercise 4.1.2(a)]. □

Assume that $\alpha,\beta\in\{1,\dots,N\}$ and let $a_{\alpha\beta}$ denote the (α,β)-entry of a matrix function $a\in L^\infty_{N\times N}(\mathbb{R})$.

Lemma 5.3. *Let $\varphi\in C_c^\infty(\mathbb{R})$. Suppose $a_l,a_r\in AP^0_{N\times N}$, $a_0\in[C_0]_{N\times N}$, and*

$$a=(1-u)a_l+ua_r+a_0.$$

Then

(a) *there exists a sequence $\{h_m\}_{m=1}^{\infty}$ such that $h_m\to+\infty$ as $m\to\infty$ and w, $\{w_m\}_{m=1}^{\infty}$ given by*

$$w:=\big((a_r)_{\alpha\beta}P+Q\big)\varphi,\quad w_m:=V_{-h_m}(a_{\alpha\beta}P+Q)V_{h_m}\varphi \tag{5.1}$$

or

$$w:=\big(\overline{(a_r)_{\alpha\beta}}P+Q\big)\varphi,\quad w_m:=V_{-h_m}(\overline{a_{\alpha\beta}}P+Q)V_{h_m}\varphi, \tag{5.2}$$

where $\alpha,\beta\in\{1,\dots,N\}$, satisfy hypotheses (i) *and* (ii) *of Lemma 4.2;*

(b) *there exists a sequence* $\{h_m\}_{m=1}^{\infty}$ *such that* $h_m \to -\infty$ *as* $m \to \infty$ *and* w, $\{w_m\}_{m=1}^{\infty}$ *given by*

$$w := \big((a_l)_{\alpha\beta} P + Q\big)\varphi, \quad w_m := V_{-h_m}(a_{\alpha\beta} P + Q) V_{h_m}\varphi$$

or

$$w := \big(\overline{(a_l)_{\beta\alpha}} P + Q\big)\varphi, \quad w_m := V_{-h_m}(\overline{a_{\beta\alpha}} P + Q) V_{h_m}\varphi,$$

where $\alpha, \beta \in \{1, \dots, N\}$, *satisfy hypotheses* (i) *and* (ii) *of Lemma* 4.2.

Proof. (a) By Lemma 5.1, there exists a sequence $\{h_m\}_{m=1}^{\infty}$ such that

$$\lim_{m\to\infty} \|a_r(\cdot + h_m) - a_r(\cdot)\|_{L^\infty_{N\times N}(\mathbb{R})} = 0. \tag{5.3}$$

Fix $\alpha, \beta \in \{1, \dots, N\}$ and consider the pair given in (5.1). It is easy to see that for $m \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$w_m(x) = (V_{-h_m} a_{\alpha\beta} V_{h_m} P\varphi)(x) + (Q\varphi)(x) = a_{\alpha\beta}(x + h_m)(P\varphi)(x) + (Q\varphi)(x). \tag{5.4}$$

From Lemma 5.2 it follows that $P\varphi, Q\varphi \in C(\mathbb{R})$ and there exists a constant $C_\varphi > 0$ such that

$$|(P\varphi)(x)| \le \frac{\widetilde{C}_\varphi}{1+|x|}, \quad |(Q\varphi)(x)| \le \frac{\widetilde{C}_\varphi}{1+|x|}, \tag{5.5}$$

where $\widetilde{C}_\varphi := (C_\varphi + \|\varphi\|_\infty)/2$. From (5.4)–(5.5) it follows that for $m \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$|w_m(x)| \le \frac{\|a_{\alpha\beta}\|_\infty \widetilde{C}_\varphi}{1+|x|}, \quad |w(x)| \le \frac{\|(a_r)_{\alpha\beta}\|_\infty \widetilde{C}_\varphi}{1+|x|}.$$

These inequalities mean that hypothesis (ii) of Lemma 4.2 holds for w, w_m given by (5.1) with $\alpha, \beta \in \{1, \dots, N\}$.

From (5.4) and the representation

$$a = (1-u)(a_l - a_r) + a_0 + a_r$$

we obtain for every $m \in \mathbb{N}$ and every $x \in \mathbb{R}$,

$$\begin{aligned} |w_m(x) - w(x)| &= |a_{\alpha\beta}(x+h_m) - (a_r)_{\alpha\beta}(x)|\,|(P\varphi)(x)| \\ &\le \big(|1 - u(x+h_m)| + |(a_0)_{\alpha\beta}(x+h_m)| + |(a_r)_{\alpha\beta}(x+h_m) - (a_r)_{\alpha\beta}(x)|\big)|(P\varphi)(x)|. \end{aligned} \tag{5.6}$$

Let $J \subset \mathbb{R}$ be a closed segment. Since $1 - u(+\infty) = 0$ and $(a_0)_{\alpha\beta}(+\infty) = 0$, we have

$$\lim_{m\to\infty} \sup_{x\in J} |1 - u(x+h_m)| = 0, \quad \lim_{k\to\infty} \sup_{x\in J} |(a_0)_{\alpha\beta}(x+h_m)| = 0. \tag{5.7}$$

From (5.3) we also have

$$\lim_{m\to\infty} \sup_{x\in J} |(a_r)_{\alpha\beta}(x+h_m) - (a_r)_{\alpha\beta}(x)| = 0. \tag{5.8}$$

The first inequality in (5.5) yields

$$\sup_{x\in J} |(P\varphi)(x)| \le \widetilde{C}_\varphi \sup_{x\in J} \frac{1}{1+|x|} < \infty. \tag{5.9}$$

From (5.6)–(5.9) we deduce that

$$\lim_{m\to\infty} \sup_{x\in J} |w_m(x) - w(x)| = 0,$$

which finishes the verification of hypothesis (i) of Lemma 4.2 for w, w_m given by (5.1) with $\alpha, \beta \in \{1, \dots, N\}$. The proof for w, w_m given by (5.2) is similar. Part (a) is proved. The proof of part (b) is analogous. □

5.2. Proof of Theorem 1.2

(a) The idea of the proof is borrowed from [34, Theorem 6.5]. Since the operator $aP+Q$ is Fredholm on $L_N^{p(\cdot)}(\mathbb{R})$, its adjoint operator $(aP+Q)^*$ is Fredholm on the dual space in view of Theorem 3.2. From Corollary 3.12 and Lemma 3.13 it follows that

$$(aP+Q)^* = Pa^*I+Q = A_1(a^*P+Q)A_2,$$

where the operators $A_1 := I+Pa^*Q$ and $A_2 := I-Qa^*P$ are invertible on $L_N^{p'(\cdot)}(\mathbb{R})$. From this equality and Lemma 3.1 we deduce that the operator a^*P+Q is Fredholm on $L_N^{p'(\cdot)}(\mathbb{R})$. Therefore, due to Theorem 3.3, the operator $A := aP+Q$ admits a left regularizer on $L_N^{p(\cdot)}(\mathbb{R})$ and the operator $A' := a^*P+Q$ admits a left regularizer on $L_N^{p'(\cdot)}(\mathbb{R})$. That is, there exist operators $B \in \mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))$, $K \in \mathcal{K}(L_N^{p(\cdot)}(\mathbb{R}))$ and $B' \in \mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))$, $K' \in \mathcal{K}(L_N^{p'(\cdot)}(\mathbb{R}))$ such that

$$BA-K=I, \quad B'A'-K'=I. \tag{5.10}$$

Since $a \in SAP_{N\times N}$, there exist $a_l, a_r \in AP_{N\times N}$ and $a_0 \in [C_0]_{N\times N}$ such that

$$a = (1-u)a_l + ua_r + a_0. \tag{5.11}$$

By the definition of AP, there exist sequences $\{a_l^{(j)}\}_{j=1}^\infty, \{a_r^{(j)}\}_{j=1}^\infty \subset AP^0_{N\times N}$ such that

$$\lim_{j\to\infty} \|a_l^{(j)} - a_l\|_{L^\infty_{N\times N}(\mathbb{R})} = 0, \quad \lim_{j\to\infty} \|a_r^{(j)} - a_r\|_{L^\infty_{N\times N}(\mathbb{R})} = 0. \tag{5.12}$$

Let $a_j := (1-u)a_l^{(j)} + ua_r^{(j)} + a_0$ and

$$A_j := a_jP+Q, \quad A_j' := a_j^*P+Q, \quad R_j := a_r^{(j)}P+Q, \quad R_j' := (a_r^{(j)})^*P+Q.$$

Put

$$J := \left[\liminf_{x\to+\infty} p(x), \limsup_{x\to+\infty} p(x)\right], \quad J' := \left[\liminf_{x\to+\infty} p'(x), \limsup_{x\to+\infty} p'(x)\right].$$

It is well known that the norm of the operator S on the standard Lebesgue spaces is calculated by

$$\|S\|_{\mathcal{B}(L^q(\mathbb{R}))} = \begin{cases} \tan\dfrac{\pi}{2q} & \text{if} \quad 1<q\le 2, \\ \cot\dfrac{\pi}{2q} & \text{if} \quad 2\le q<\infty \end{cases}$$

(see, e.g., [14, Chap. 13, Theorem 1.3]). Hence

$$\sup_{q\in J\cup J'} \max\left\{\|P\|_{L^q_N(\mathbb{R}))}, \|Q\|_{L^q_N(\mathbb{R}))}\right\} =: M < \infty.$$

If we denote $R := a_rP+Q$ and $R' := P+a_r^*Q$, then

$$\sup_{q\in J} \|R-R_j\|_{\mathcal{B}(L^q_N(\mathbb{R}))} \le C_NM\|a_r - a_r^{(j)}\|_{L^\infty_{N\times N}(\mathbb{R})}, \tag{5.13}$$

$$\sup_{q'\in J'} \|R'-R_j'\|_{\mathcal{B}(L^{q'}_N(\mathbb{R}))} \le C_NM\|a_r - a_r^{(j)}\|_{L^\infty_{N\times N}(\mathbb{R})}, \tag{5.14}$$

where the constant $C_N > 0$ depends only on N. From (5.11)–(5.12) it follows that

$$\|A-A_j\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))} < \frac{1}{2\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}}, \tag{5.15}$$

$$\|A'-A_j'\|_{\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))} < \frac{1}{2\|B'\|_{\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))}} \tag{5.16}$$

for sufficiently large j. Further, from (5.12)–(5.14) we also deduce that

$$\sup_{q\in J}\|R-R_j\|_{\mathcal{B}(L_N^q(\mathbb{R}))}<\frac{1}{2\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}}, \tag{5.17}$$

$$\sup_{q'\in J'}\|R'-R'_j\|_{\mathcal{B}(L_N^{q'}(\mathbb{R}))}<\frac{1}{2\|B'\|_{\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))}} \tag{5.18}$$

for sufficiently large j. Fix j such that all inequalities (5.15)–(5.18) are fulfilled simultaneously.

From the first equality in (5.10) and (5.15) it follows that for every $f\in L_N^{p(\cdot)}(\mathbb{R})$,

$$\begin{aligned}\|f\|_{L_N^{p(\cdot)}(\mathbb{R})} &\le \|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|Af\|_{L_N^{p(\cdot)}(\mathbb{R})}+\|Kf\|_{L_N^{p(\cdot)}(\mathbb{R})}\\ &\le \|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\big(\|A_jf\|_{L_N^{p(\cdot)}(\mathbb{R})}+\|Af-A_jf\|_{L_N^{p(\cdot)}(\mathbb{R})}\big)+\|Kf\|_{L_N^{p(\cdot)}(\mathbb{R})}\\ &\le \|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|A_jf\|_{L_N^{p(\cdot)}(\mathbb{R})}+\frac{1}{2}\|f\|_{L_N^{p(\cdot)}(\mathbb{R})}+\|Kf\|_{L_N^{p(\cdot)}(\mathbb{R})}.\end{aligned}$$

Hence for all $f\in L_N^{p(\cdot)}(\mathbb{R})$,

$$\|f\|_{L_N^{p(\cdot)}(\mathbb{R})}\le 2\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|A_jf\|_{L_N^{p(\cdot)}(\mathbb{R})}+2\|Kf\|_{L_N^{p(\cdot)}(\mathbb{R})}. \tag{5.19}$$

Analogously, from the second equality in (5.10) and (5.16) we obtain for $g\in L_N^{p'(\cdot)}(\mathbb{R})$,

$$\|g\|_{L_N^{p'(\cdot)}(\mathbb{R})}\le 2\|B'\|_{\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))}\|A'_jg\|_{L_N^{p'(\cdot)}(\mathbb{R})}+2\|K'g\|_{L_N^{p'(\cdot)}(\mathbb{R})}. \tag{5.20}$$

Let ψ_n be as in Lemma 3.15. It is clear that $\Psi_n:=\operatorname{diag}\{\psi_nI,\dots,\psi_nI\}$ is an idempotent, that is, $\Psi_n^2=\Psi_n$. By Lemma 3.15(b), there exists an $n\in\mathbb{N}$ such that

$$\|K\Psi_n\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\le\frac{1}{4},\quad \|K'\Psi_n\|_{\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))}\le\frac{1}{4}.$$

Hence for all $f\in L_N^{p(\cdot)}(\mathbb{R})$,

$$\|K\Psi_nf\|_{L_N^{p(\cdot)}(\mathbb{R})}=\|K\Psi_n^2f\|_{L_N^{p(\cdot)}(\mathbb{R})}\le\|K\Psi_n\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|\Psi_nf\|_{L_N^{p(\cdot)}(\mathbb{R})}\le\frac{1}{4}\|\Psi_nf\|_{L_N^{p(\cdot)}(\mathbb{R})}, \tag{5.21}$$

and similarly

$$\|K'\Psi_ng\|_{L_N^{p'(\cdot)}(\mathbb{R})}\le\frac{1}{4}\|\Psi_ng\|_{L_N^{p'(\cdot)}(\mathbb{R})}. \tag{5.22}$$

From (5.19) and (5.21) it follows that for all $f\in L_N^{p(\cdot)}(\mathbb{R})$,

$$\|\Psi_nf\|_{L_N^{p(\cdot)}(\mathbb{R})}\le 4\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|A_j\Psi_nf\|_{L_N^{p(\cdot)}(\mathbb{R})} \tag{5.23}$$

In the same way, from (5.20) and (5.22) we obtain for all $g\in L_N^{p'(\cdot)}(\mathbb{R})$,

$$\|\Psi_ng\|_{L_N^{p'(\cdot)}(\mathbb{R})}\le 4\|B'\|_{\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))}\|A'_j\Psi_ng\|_{L_N^{p'(\cdot)}(\mathbb{R})}. \tag{5.24}$$

Let $\varphi\in[C_c^\infty(\mathbb{R})]_N$. In view of Lemma 5.3(a), there exists a sequence $\{h_m\}_{m=1}^\infty$ such that $h_m\to+\infty$ as $m\to\infty$ and each of the functions given by

$$w_{\alpha\beta}:=\big((a_r^{(j)})_{\alpha\beta}P+Q\big)\varphi_\beta,\quad (w_{\alpha\beta})_m:=V_{-h_m}((a_j)_{\alpha\beta}P+Q)V_{h_m}\varphi_\beta$$

and

$$w'_{\alpha\beta}:=\big(\overline{(a_r^{(j)})_{\beta\alpha}}P+Q\big)\varphi_\beta,\quad (w_{\alpha\beta})'_m:=V_{-h_m}(\overline{(a_j)_{\beta\alpha}}P+Q)V_{h_m}\varphi_\beta$$

for $\alpha, \beta \in \{1, \dots, N\}$ satisfies hypotheses (i) and (ii) of Lemma 4.2.

For $h \in \mathbb{R}$, let the translation operator V_h be defined on $L_N^{p(\cdot)}(\mathbb{R})$ and on $L_N^{p'(\cdot)}(\mathbb{R})$ elementwise (although it may be unbounded on these spaces). It is easy to see that there exists an $m_0 \in \mathbb{N}$ such that

$$\Psi_n V_{h_m} \varphi = V_{h_m} \varphi \quad \text{for all} \quad m \geq m_0. \tag{5.25}$$

Then from (5.23) and (5.25) it follows that for all $\varphi \in [C_c^\infty(\mathbb{R})]_N$ and all $m \geq m_0$,

$$\begin{aligned} \|V_{h_m}\varphi\|_{L_N^{p(\cdot)}(\mathbb{R})} &= \|\Psi_n V_{h_m}\varphi\|_{L_N^{p(\cdot)}(\mathbb{R})} \\ &\leq 4\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|A_j\Psi_n V_{h_m}\varphi\|_{L_N^{p(\cdot)}(\mathbb{R})} \\ &= 4\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|A_j V_{h_m}\varphi\|_{L_N^{p(\cdot)}(\mathbb{R})} \\ &= 4\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|V_{h_m}(V_{-h_m}A_j V_{h_m}\varphi)\|_{L_N^{p(\cdot)}(\mathbb{R})}. \end{aligned} \tag{5.26}$$

Analogously, from (5.24) and (5.25) we get for all $\varphi \in [C_c^\infty(\mathbb{R})]_N$ and all $m \geq m_0$,

$$\|V_{h_m}\varphi\|_{L_N^{p'(\cdot)}(\mathbb{R})} \leq 4\|B'\|_{\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))}\|V_{h_m}(V_{-h_m}A_j' V_{h_m}\varphi)\|_{L_N^{p'(\cdot)}(\mathbb{R})}. \tag{5.27}$$

Since the sequence $\{p(h_m)\}_{m=1}^\infty$ is bounded, $p_- \leq p(h_m) \leq p_+$ for all $m \in \mathbb{N}$, there exists its convergent subsequence $\{p(h_{m_k})\}_{k=1}^\infty$. Let

$$q_r := \lim_{k\to\infty} p(h_{m_k}).$$

It is clear that $q_r \in J$. Taking into account (1.1) we also see that

$$\lim_{k\to\infty} p'(h_{m_k}) = q_r/(q_r - 1) =: q_r' \in J'.$$

Applying Lemma 4.2 to

$$w_{\alpha\beta} := \big((a_r^{(j)})_{\alpha\beta}P + Q\big)\varphi_\beta, \quad (w_{\alpha\beta})_{m_k} := V_{-h_{m_k}}\big((a_j)_{\alpha\beta}P + Q\big)V_{h_{m_k}}\varphi_\beta$$

with $\alpha, \beta \in \{1, \dots, N\}$, we obtain

$$\lim_{k\to\infty}\|V_{h_{m_k}}(V_{-h_{m_k}}(A_j)_{\alpha\beta}V_{h_{m_k}}\varphi_\beta\|_{p(\cdot)} = \lim_{k\to\infty}\|V_{h_{m_k}}(w_{\alpha\beta})_{m_k}\|_{p(\cdot)} = \|w_{\alpha,\beta}\|_{q_r} = \|(R_j)_{\alpha\beta}\varphi_\beta\|_{q_r}.$$

Then

$$\lim_{k\to\infty}\|V_{h_{m_k}}(V_{-h_{m_k}}A_j V_{h_{m_k}}\varphi)\|_{L_N^{p(\cdot)}(\mathbb{R})} = \|R_j\varphi\|_{L_N^{q_r}(\mathbb{R})}. \tag{5.28}$$

Analogously, applying Lemma 4.2 to

$$w_{\alpha\beta}' := \big(\overline{(a_r^{(j)})_{\beta\alpha}}P + Q\big)\varphi_\beta, \quad (w_{\alpha\beta})_{m_k}' := V_{-h_{m_k}}(\overline{(a_j)_{\beta\alpha}}P + Q)V_{h_{m_k}}\varphi_\beta$$

with $\alpha, \beta \in \{1, \dots, N\}$ on the dual space, we get

$$\lim_{k\to\infty}\|V_{h_{m_k}}(V_{-h_{m_k}}A_j' V_{h_{m_k}}\varphi)\|_{L_N^{p'(\cdot)}(\mathbb{R})} = \|R_j'\varphi\|_{L_N^{q_r'}(\mathbb{R})}. \tag{5.29}$$

Finally, applying Lemma 4.2 to the constant sequences $w_k = \varphi_\beta$ and $w = \varphi_\beta$ for all $\beta \in \{1, \dots, N\}$, we get

$$\lim_{k\to\infty}\|V_{h_{m_k}}\varphi\|_{L_N^{p(\cdot)}(\mathbb{R})} = \|\varphi\|_{L_N^{q_r}(\mathbb{R})}, \quad \lim_{k\to\infty}\|V_{h_{m_k}}\varphi\|_{L_N^{p'(\cdot)}(\mathbb{R})} = \|\varphi\|_{L_N^{q_r'}(\mathbb{R})}. \tag{5.30}$$

Inequalities (5.26) and (5.27), in particular, imply that for all $k \in \mathbb{N}$ and $\varphi \in [C_c^\infty(\mathbb{R})]_N$,

$$\begin{aligned} \|V_{h_{m_k}}\varphi\|_{L_N^{p(\cdot)}(\mathbb{R})} &\leq 4\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|V_{h_{m_k}}(V_{-h_{m_k}}A_j V_{h_{m_k}}\varphi)\|_{L_N^{p(\cdot)}(\mathbb{R})}, \\ \|V_{h_{m_k}}\varphi\|_{L_N^{p'(\cdot)}(\mathbb{R})} &\leq 4\|B'\|_{\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))}\|V_{h_{m_k}}(V_{-h_{m_k}}A_j' V_{h_{m_k}}\varphi)\|_{L_N^{p'(\cdot)}(\mathbb{R})}. \end{aligned}$$

Passing in these inequalities to the limit as $k \to \infty$ and taking into account equalities (5.28)–(5.30), we obtain for all $\varphi \in [C_c^\infty(\mathbb{R})]_N$,

$$\|\varphi\|_{L_N^{q_r}(\mathbb{R})} \le 4\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|R_j\varphi\|_{L_N^{q_r}(\mathbb{R})}, \tag{5.31}$$

$$\|\varphi\|_{L_N^{q_r'}(\mathbb{R})} \le 4\|B'\|_{\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))}\|R_j'\varphi\|_{L_N^{q_r'}(\mathbb{R})}. \tag{5.32}$$

From inequalities (5.17) and (5.31) we obtain

$$\begin{aligned}\|\varphi\|_{L_N^{q_r}(\mathbb{R})} \le& 4\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|R\varphi\|_{L_N^{q_r}(\mathbb{R})} + 4\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|R-R_j\|_{\mathcal{B}(L_N^{q_r}(\mathbb{R}))}\|\varphi\|_{L_N^{q_r}(\mathbb{R})}\\ \le& 4\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|R\varphi\|_{L_N^{q_r}(\mathbb{R})} + \frac{1}{2}\|\varphi\|_{L_N^{q_r}(\mathbb{R})}.\end{aligned}$$

Hence, for all $\varphi \in [C_c^\infty(\mathbb{R})]_N$,

$$\|\varphi\|_{L_N^{q_r}(\mathbb{R})} \le 8\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|R\varphi\|_{L_N^{q_r}(\mathbb{R})}. \tag{5.33}$$

Let $f \in L_N^{q_r}(\mathbb{R})$ and $\{\varphi_k\}_{k=1}^\infty \subset [C_c^\infty(\mathbb{R})]_N$ be a sequence such that

$$\lim_{k\to\infty}\|f-\varphi_k\|_{L_N^{q_r}(\mathbb{R})} = 0.$$

From this equality and (5.33) it follows that

$$\|f\|_{L_N^{q_r}(\mathbb{R})} = \lim_{k\to\infty}\|\varphi_k\|_{L_N^{q_r}(\mathbb{R})} \le 8\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\lim_{k\to\infty}\|R\varphi_k\|_{L_N^{q_r}(\mathbb{R})} = 8\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}\|Rf\|_{L_N^{q_r}(\mathbb{R})}.$$

Therefore

$$0 < \frac{1}{8\|B\|_{\mathcal{B}(L_N^{p(\cdot)}(\mathbb{R}))}} \le \mathcal{J}(R; L_N^{q_r}(\mathbb{R})). \tag{5.34}$$

Arguing analogously and starting with (5.18) and (5.32), we obtain

$$0 < \frac{1}{8\|B'\|_{\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))}} \le \mathcal{J}(R'; L_N^{q_r'}(\mathbb{R})). \tag{5.35}$$

From Corollary 3.12 and Lemma 3.13 we obtain

$$R^* = (a_rP+Q)^* = Pa_r^*I + Q = A_3R'A_4,$$

where $A_3 := I + Pa_r^*Q$ and $A_4 := I - Qa_r^*P$ are invertible on $L_N^{q_r'}(\mathbb{R})$. From this equality, Lemma 3.5 and Theorem 3.6 it follows that

$$\mathcal{J}(R^*; L_N^{q_r'}(\mathbb{R})) \ge \mathcal{J}(A_3; L_N^{q_r'}(\mathbb{R}))\cdot\mathcal{J}(R'; L_N^{q_r'}(\mathbb{R}))\cdot\mathcal{J}(A_4; L_N^{q_r'}(\mathbb{R})) = \frac{\mathcal{J}(R'; L_N^{q_r'}(\mathbb{R}))}{\|A_3^{-1}\|_{\mathcal{B}(L_N^{q_r'}(\mathbb{R}))}\|A_4^{-1}\|_{\mathcal{B}(L_N^{q_r'}(\mathbb{R}))}}. \tag{5.36}$$

From (3.4) we see that

$$\begin{aligned}\|A_3^{-1}\|_{\mathcal{B}(L_N^{q_r'}(\mathbb{R}))} &= \|I - Pa_r^*Q\|_{\mathcal{B}(L_N^{q_r'}(\mathbb{R}))}\\ &\le 1 + \|P\|_{\mathcal{B}(L_N^{q_r'}(\mathbb{R}))}\|a_r^*I\|_{\mathcal{B}(L_N^{q_r'}(\mathbb{R}))}\|Q\|_{\mathcal{B}(L_N^{q_r'}(\mathbb{R}))}\\ &\le 1 + C_N\|a_r\|_{L_{N\times N}^\infty(\mathbb{R})}M^2 \end{aligned}\tag{5.37}$$

and analogously

$$\|A_4^{-1}\|_{\mathcal{B}(L_N^{q_r'}(\mathbb{R}))} \le 1 + C_N\|a_r\|_{L_{N\times N}^\infty(\mathbb{R})}M^2. \tag{5.38}$$

Combining (5.35)–(5.38), we arrive at

$$\mathcal{J}(R^*; L_N^{q_r'}(\mathbb{R})) \geq \frac{\mathcal{J}(R'; L_N^{q_r'}(\mathbb{R}))}{\left(1 + C_N \|a_r\|_{L^\infty_{N\times N}(\mathbb{R})} M^2\right)^2} \geq \frac{\left(8\|B'\|_{\mathcal{B}(L_N^{p'(\cdot)}(\mathbb{R}))}\right)^{-1}}{\left(1 + C_N \|a_r\|_{L^\infty_{N\times N}(\mathbb{R})} M^2\right)^2} =: M_1 > 0.$$

From this inequality and Lemma 3.4 we conclude that

$$\mathcal{Q}(R; L_N^{q_r}(\mathbb{R})) \geq M_1 > 0. \tag{5.39}$$

Finally, inequalities (5.34), (5.39) and Theorem 3.6 imply that the operator $R = a_r P + Q$ is invertible on the standard Lebesgue space $L_N^{q_r}(\mathbb{R})$. □

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